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A REMARK ON A THEOREM OF SHUB AND SULLIVAN†

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INTRODUCTION

DIFFERENTIABILITY hypotheses sometimes play a delicate role in (asymptotic) fixed point theory. One of the most prominent examples for this fact is the Schauder conjecture: *Let $U \subset E$ be an open and convex subset of a Banach space E, and let $T:\bar{U} \to U$ be a continuous map such that T^n is a compact map for some $n \in N$. Then (?) T has a fixed point.* The conjecture is true if the map T is C^1[4, 8], but up to now there is no answer in the case where T is only continuous. Another surprising example is provided by a theorem of Shub and Sullivan[6] on the Lefschetz fixed point formula.

THEOREM (SHUB-SULLIVAN). *Let $f:M \to M$ be a C^1-endomorphism of a compact differentiable manifold M. If the sequence of Lefschetz number $\{L(f^n)\}_{n \in N}$ is not bounded then the set of periodic points of f is infinite.*

This theorem is not true if the map f is only continuous, see[6]. That the theorem does not hold even for simplicial maps is demonstrated in the following example.

Example. Let $f:S^2 \to S^2$ be the following simplicial map of the sphere (a vertex labelled with a number i indicates that it is mapped into the vertex $[i]$):

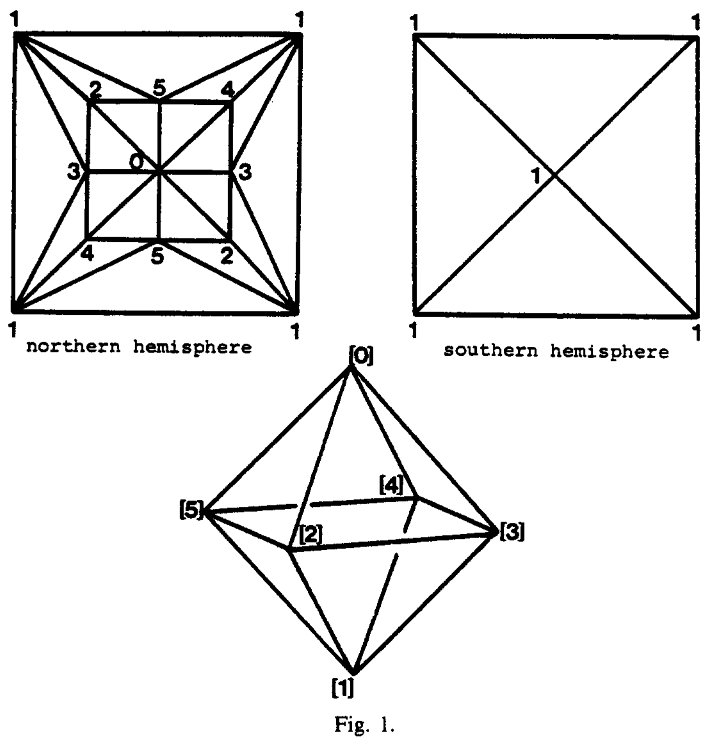

Fig. 1.

Then $L(f^n) = 1 + 2^n$, but [0] and [1] are the only periodic points of f.

On the other side there are examples where the theorem holds also for continuous

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mappings, for example, if on the manifold M the modulus of the Lefschetz number gives a lower bound for the Nielsen number[2]. Such examples are S^1[2] and the n-torus T^n[1]. (Halpern recently proved a 'non C^1-version' of the theorem of the Klein bottle, (see[3], p. 507).)

In this note we will prove a version of the Shub–Sullivan theorem for a special class of simplicial mappings of a compact polyhedron P. The maps under consideration are of type $(P, \tau')\to(P, \tau)$ where τ' is a subdivision of the triangulation τ of P. As the example above shows the subdivision τ' has to satisfy conditions which prevent "non-differentiable" cusps for the iterates of f (like the point $[O]$ in the example above). It turns out that in this context the Shub–Sullivan theorem can be sharpened in such a way that there exist infinitely many periodic points with distinct periods.

The Shub–Sullivan theorem was brought to my attention by Heinz–Otto Peitgen. Conversations with him and with Gentscho Skordev were of great benefit for me.

Results. In order to prove our simplicial version of the Shub–Sullivan result we will use the Hopf trace formula to reduce the situation to the following proposition.

PROPOSITION: *Let S be a closed n-simplex, and let S_1, $S_2 \subset S$ be closed n-simplices whose intersection is either empty or a common face. Suppose the following condition is satisfied: (C) For all j, $0 \le j < n$, there is no common j-dimensional face of S_1 and S_2 which is contained in a j-dimensional face of S.*

If $g: S_1 \cup S \to S$ is a continuous map which maps S_1 and S_2 affine linearly onto S, then for every prime number $p \in N$ there exists a periodic point of period p for g.

Proof. Let $g_i = g|_{S_i}: S_i \to S$, $i = 1, 2$. Then g_i is a homeomorphism, and the Brouwer fixed point theorem applies to g_i^{-1}: $S \to S_i \subset S$. Hence g has a fixed point. (In fact, there are two fixed points for g due to condition (C).)

We find recursively closed n-simplices B_j, $0 \leqq j \leqq p$, with $B_p = S$, $B_{p-1} = S_2$, $B_i \subset S_1$, $0 \leqq i \leqq p-2$, such that

$$g(B_j, \partial B_j) = (B_{j+1}, \partial B_{j+1})\text{(affine linearly)}.$$

Hence g^p has a fixed point $x_o \in B_0$.

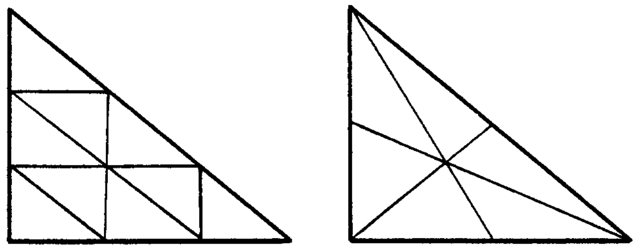
In order to verify that x_o is a periodic point of period p we only need to consider the following situation:

$$x_o \in \partial B_0,\ S_i \cap S_2 \neq \emptyset,\ S_i \cap \partial S \neq \emptyset \quad i = 1, 2.$$

(If $S_1 \cap S_2 = \emptyset$, then $g^{p-1}(x_o) \neq g^j(x_o), 0 \leqq j < p-1$, and therefore x_o has period p. If $S_1 \cap \partial S = \emptyset$, then $x_o \in \operatorname{int} B_0$ and therefore $g^{p-1}(x_o) \neq g^j(x_0)$, $0 \leqq j < p-1$, because g is affine linear.)

Assume that x_o is not a periodic point of period p. Because p is prime, x_o must be a fixed point of g. We claim that this is impossible because of condition (C). Let $R \subset \partial S$ be the face of S which supports x_o (i.e. R is the face of S such that $x_o \in \operatorname{int} R$) and set $d = \dim R$. If $d = 0$ then x_o is a common vertex of S_1, S_2 and S which is excluded by condition (C). If $0 < d < n$ let R_2 be the face of S_2 with $g(R_2) = R$. Since $x_o \in \operatorname{int} R \cap \operatorname{int} R_2$ it follows $R_2 \subset R \subset \partial S$. But $x_o \in S_1 \cap S_2$(= a common face) and $x_o \in \operatorname{int} R_2$, thus R_2 is also a face of S_1 which again is excluded by condition (C).

In view of the proposition above we call a subdivision τ' of a triangulation τ *regular*, provided for all $S \in \tau$ and for all S_1, $S_2 \in \tau'$ with $S_i \subset S$ and $\dim S_i = \dim S$, $i = 1, 2$, condition (C) is satisfied.



regular not regular

Fig. 2.

COROLLARY. *Let (P, τ) be a compact polyhedron with triangulation τ, and let τ' be a regular subdivision of τ. If $f:(P, \tau')\to(P, \tau)$ is a simplicial map and if the sequence of Lefschetz numbers $\{L(f^n)\}_{n\in N}$ is unbounded, then f has infinitely many periodic points with distinct periods.*

Proof. Let τ_i, $i = 1, 2, \ldots$ be subdivisions of $\tau_o := \tau'$ such that f is a simplicial mapping

$$(P, \tau_i)\to(P, \tau_{i-1}).$$

The following diagram is commutative

$$\begin{array}{ccc} C_*(\tau') & \xrightarrow{f_*} & C_*(\tau) \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle b} \quad i = 1, 2, \ldots \\ C_*(\tau_i) & \xrightarrow[f_*]{} & C_*(\tau_{i-1}). \end{array}$$

Here C_* denotes the (oriented) simplicial chain complex, f_* the induced mapping and b die subdivision map (rational coefficients). Let $h:C_*(\tau')\to C_*(\tau')$ be the composition

$$h = b\circ f_*:\ C_*(\tau')\to C_*(\tau)\to C_*(\tau').$$

Then

$$h^i = (f^i)_*\circ b$$

with

$$b:C_*(\tau')\to C_*(\tau_i),\ (f^i)_*:C_*(\tau_i)\to C_*(\tau_o).$$

Hence, by the Hopf trace formula we obtain

$$L(f^i) = (h^i) = \sum_{n=0}^{\dim P} (-1)^n \operatorname{trace}(h^i|_{C_n}: C_n(\tau')\to C_n(\tau')).$$

By our assumption there exists $i\in N$ and $n\in N$, $1\le n\le \dim P$, such that

$$|\operatorname{trace}(h^i|_{C_n}: C_n(\tau')\to C_n(\tau'))| > \dim C_n(\tau').$$

But this implies that we can find an n-simplex $S\in C_n(\tau')$ and n-dimensional subsimplices S_1, $S_2\in C_n(\tau_i)$ which are mapped by f^i affine linearly onto S. Moreover, because τ' is a regular subdivision of τ, the simplices S, S_1, S_2 satisfy condition (C). Thus, the proposition above yields the result.

Using an appropriate simplicial definition of the fixed point index we can also prove a localized version of the corollary above (corresponding to the proposition in ([6, p. 189])).

THEOREM. *Let (P, τ) be a compact polyhedron with triangulation τ and let τ' be a regular subdivision of τ. Let $U \subset P$ be an open subset of P such that $\bar{U}$ can be triangulated by τ'.*

Let $f:(\bar{U}, \tau') \to (P, \tau)$ be a simplicial map with an isolated fixed point $x_o \in U$.

If x_o is an isolated fixed point for all f^i, $i \in N$, then the sequence of fixed point indices $\{\mathrm{ind}(f^i, x_o)\}_{i \in N}$ is bounded.

Proof. Let $V_i := f^{-1}(\bar{U})$, $i \in N$. We find triangulations τ_i of V_i such that

$$f^i:(V_{i-1}, \tau_{i-1}) \to (\bar{U}, \tau') \to (P, \tau), i \geq 2$$

is a simplicial map. Since f^i is simplicial and since $x_o \in V_{i-1}$ is an isolated fixed point of f^i we can compute the fixed point index $\mathrm{ind}(f^i, x_o)$ as follows:

$$\mathrm{ind}(f^i, x_o) = \sum_{\substack{\sigma \in \tau_{i-1} \\ x_o \in \sigma}} (-1)^{\dim \sigma} \text{ or } (\sigma)$$

with

$$\mathrm{or}(\sigma) = \begin{cases} 1(-1), f^i(\sigma) \supset \sigma \text{ and } f^i|\sigma \text{ orientation preserving (reversing)} \\ 0, \quad \text{otherwise.} \end{cases}$$

This is a consequence of O'Neill's definition of fixed point index[5], see also[7].

Therefore, assuming that $\{\mathrm{ind}(f^i, x_o)\}_{i \in N}$ is not bounded we find $n \in N$ and simplices σ_1, $\sigma_2 \in \tau_{n-1}$ ($\dim \sigma_1 = \dim \sigma_2$) such that $f^n(\sigma_1) = f^n(\sigma_2) =: \sigma \supset \sigma_1 \cup \sigma_2$, $\sigma \in \tau$.

Because of condition (C) f^n has fixed points $a_1 \in \sigma_1$ and $a_2 \in \sigma_2$, $a_1 \neq a_2$. Since $f^n|_{\sigma_i}$, $i = 1$, 2, is affine linear this implies that the fixed point x_o of f^n cannot be isolated and we are done.

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